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REPRESENTATIONS OF KRONECKER POWERS OF ORTHOGONAL TENSORS WITH APPLICATIONS TO MATERIAL SYMMETRY

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Abstract—Two explicit representations are obtained for the Kronecker powers of orthogonal second-order tensors. The derivations rely on the mathematical properties of Kronecker products and on classical parametrizations of orthogonal tensors. These representations are subsequently employed in the systematic construction of structural tensors and in the analysis of corotational rates of tensor functions.

1. INTRODUCTION

Consider a k-th order tensor $\mathbf{T}$ in the three-dimensional Euclidean space E^3, having $T_{j_1j_2\ldots j_k}$ and $\bar{T}_{i_1i_2\ldots i_k}$ as its components with reference to fixed orthonormal bases $\{\mathbf{e}_i\}$ and $\{\bar{\mathbf{e}}_i\}$, respectively. Let the bases vectors in the two systems be related by the orthogonal transformation $\bar{\mathbf{e}}_i = Q_{ij}\mathbf{e}_j$, where Q_{ij} are the components of an orthogonal tensor $\mathbf{Q}$. Then, the components of $\mathbf{T}$ are related according to

$$\bar{T}_{i_1i_2\ldots i_k} = Q_{i_1j_1}Q_{i_2j_2}\ldots Q_{i_kj_k}T_{j_1j_2\ldots j_k}. \tag{1}$$

The quantities $Q_{i_1j_1}Q_{i_2j_2}\ldots Q_{i_kj_k}$ are components of the k-th Kronecker power of the orthogonal tensor $\mathbf{Q}$, denoted here as $\mathbb{Q}_k$. Recalling that the space of k-th order tensors may be regarded as a 3^k-dimensional Euclidean vector space, it follows that $\mathbb{Q}_k$ may be interpreted as a linear transformation on this vector space. In fact, it is established later in the article that this is an orthogonal transformation. Given that second-order orthogonal tensors admit various representations, a question arises as to whether analogous representations can be obtained for $\mathbb{Q}_k$. A polynomial expansion of $\mathbb{Q}_k$ for the special case $k = 2$ has been derived by Podio-Guidugli and Virga (1987), while Mehrabadi *et al.* (1995) have also deduced an associated exponential representation by exploiting the properties of fourth-order tensors. The main purpose of this article is to provide completely general exponential, as well as polynomial representations of $\mathbb{Q}_k$.

The motivation for this work comes from the study of material symmetry, where transformations as in eqn (1) play an important role in the precise characterization of material behavior. It is expected that closed-form expressions for $\mathbb{Q}_k$ will facilitate the analytical and computational investigation of anisotropic continua. As a representative application, the systematic construction of *structural* (or *anisotropic*) tensors will be discussed. The structural tensors are k-th order tensors which remain invariant under transformation of the type (1). These tensors play an important role in the formulation of anisotropic response functions. Indeed, it has been shown in Boehler (1979) and Liu (1982) that every anisotropic function can be derived from isotropic functions with the inclusions of appropriately defined structural tensors. Zheng and Spender (1993a) have recently noted that the structural tensors can be represented as linear combinations of the eigenvectors of $\mathbb{Q}_k$ corresponding to unit eigenvalues, thus opening a mathematically transparent way of

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constructing structural tensors. This problem is revisited in light of the new representations. The proposed exponential representation of $\mathbb{Q}_k$ is also important in the study of isotropic tensor functions. In particular, it is shown that the exponential representation, in conjunction with certain geometric techniques, provides a novel and general framework for interpreting such functions. The detailed procedure is further illustrated by an example concerning the corotational rates of isotropic tensor functions.

The organization of this article is as follows: relevant properties of the Kronecker products are summarized in Section 2. A general exponential representation of $\mathbb{Q}_k$ is derived in Section 3, while a polynomial counterpart is obtained in Section 4. The article is concluded with two applications to material symmetry presented in Section 5.

2. PROPERTIES OF KRONECKER PRODUCTS

Let E^m and E^n be m- and n-dimensional Euclidean vector spaces, and consider two tensors $\mathbf{A}: E^m \mapsto E^m$ and $\mathbf{B}: E^n \mapsto E^n$. The Kronecker product $\mathbf{A} \boxtimes \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is defined as a linear transformation on the product space $E^m \times E^n$, such that

$$\mathbf{A} \boxtimes \mathbf{B}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{A}\mathbf{a}) \otimes (\mathbf{B}\mathbf{b}), \tag{2}$$

for any vectors $\mathbf{a} \in E^m$ and $\mathbf{b} \in E^n$. The remainder of this section contains a brief review of certain properties of Kronecker products. For an original account, see, e.g., the classical monograph of Murnaghan (1938).

The transpose of the Kronecker product between $\mathbf{A}$ and $\mathbf{B}$ is defined such that

$$(\mathbf{a} \otimes \mathbf{b}) \cdot [(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{c} \otimes \mathbf{d})] = [(\mathbf{A} \boxtimes \mathbf{B})^T(\mathbf{a} \otimes \mathbf{b})] \cdot (\mathbf{c} \otimes \mathbf{d}), \tag{3}$$

for all vectors $\mathbf{a}$, $\mathbf{c}$ in E^m and $\mathbf{b}$, $\mathbf{d}$ in E^n. In the above equation, "$\cdot$" denotes the usual inner product between vectors. The following properties can be verified directly using (2) and (3):

$$\begin{aligned} \mathbf{A} \boxtimes \mathbf{0} &= \mathbf{0} \boxtimes \mathbf{B} = \mathbb{O}, \\ (\mathbf{A}+\mathbf{C}) \boxtimes \mathbf{B} &= \mathbf{A} \boxtimes \mathbf{B} + \mathbf{C} \boxtimes \mathbf{B}, \\ (\alpha\mathbf{A}) \boxtimes (\beta\mathbf{B}) &= \alpha\beta(\mathbf{A} \boxtimes \mathbf{B}), \\ (\mathbf{A} \boxtimes \mathbf{B})(\mathbf{C} \boxtimes \mathbf{D}) &= (\mathbf{A}\mathbf{C}) \boxtimes (\mathbf{B}\mathbf{D}), \\ (\mathbf{A} \boxtimes \mathbf{B})^T &= \mathbf{A}^T \boxtimes \mathbf{B}^T, \end{aligned} \tag{4}$$

for any tensors $\mathbf{A}$, $\mathbf{C}$ on E^m, for any tensors $\mathbf{B}$, $\mathbf{D}$ on E^n, and for any real numbers α and β. In (4a), $\mathbf{0}$ denotes the zero tensor on E^m or E^n, and $\mathbb{O}$ the zero tensor on $E^m \times E^n$. If the tensors $\mathbf{A}$ and $\mathbf{B}$ are non-singular, then the inverse of the Kronecker product $\mathbf{A} \boxtimes \mathbf{B}$ is defined as

$$(\mathbf{A} \boxtimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \boxtimes \mathbf{B}^{-1}, \tag{5}$$

so that from (5)

$$(\mathbf{A} \boxtimes \mathbf{B})^{-1}(\mathbf{A} \boxtimes \mathbf{B}) = (\mathbf{A} \boxtimes \mathbf{B})(\mathbf{A} \boxtimes \mathbf{B})^{-1} = \mathbf{I}_m \boxtimes \mathbf{I}_n,$$

where $\mathbf{I}_m$ and $\mathbf{I}_n$ are the identity tensors on E^m and E^n, respectively.

Any three tensors **A**, **B** and **E** on E^m, E^n and E^p, respectively, satisfy the associativity property

$$(\mathbf{A} \boxtimes \mathbf{B}) \boxtimes \mathbf{E} = \mathbf{A} \boxtimes (\mathbf{B} \boxtimes \mathbf{E}). \tag{6}$$

Taking into account (6), the k-th order Kronecker power of a tensor **A** is simply written as

$$\mathbb{A}_k = \underbrace{\mathbf{A} \boxtimes \mathbf{A} \boxtimes \cdots \boxtimes \mathbf{A}}_{k \text{ terms}}. \tag{7}$$

Letting $\mathbf{I}_2 \equiv \mathbf{I}$ be the second-order identity tensor on E^3, it can be immediately verified using (2) and (7) that $\mathbb{I}_k$, defined as

$$\mathbb{I}_k = \underbrace{\mathbf{I} \boxtimes \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I}}_{k \text{ terms}},$$

is the identity tensor of order $2k$. With reference to (4e), it is noted here that the Kronecker power $\mathbb{Q}_k$ of a second-order orthogonal tensor **Q** is itself an orthogonal tensor in the sense that

$$\mathbb{Q}_k^T\mathbb{Q}_k = \underbrace{(\mathbf{Q} \boxtimes \mathbf{Q} \boxtimes \cdots \boxtimes \mathbf{Q})}_{k \text{ terms}}{}^T \underbrace{(\mathbf{Q} \boxtimes \mathbf{Q} \boxtimes \cdots \boxtimes \mathbf{Q})}_{k \text{ terms}}$$

$$= \underbrace{(\mathbf{Q}^T\mathbf{Q}) \boxtimes (\mathbf{Q}^T\mathbf{Q}) \boxtimes \cdots \boxtimes (\mathbf{Q}^T\mathbf{Q})}_{k \text{ terms}} = \mathbb{I}_k.$$

In order to explore the spectral properties of Kronecker products, consider a real polynomial $K(x, y) = \Sigma_{p=0}^{m} \Sigma_{q=0}^{n} \alpha_{pq} x^p y^q$ of order m in x and n in y, where α_{pq} are given scalar coefficients, and let the associated Kronecker product $\mathbb{K}$ be defined as

$$\mathbb{K}(\mathbf{A}, \mathbf{B}) = \sum_{p=0}^{m} \sum_{q=0}^{n} \alpha_{pq} \mathbf{A}^p \boxtimes \mathbf{B}^q. \tag{8}$$

In the above equation, it is understood that $\mathbf{A}^0$ and $\mathbf{B}^0$ correspond to $\mathbf{I}_m$ and $\mathbf{I}_n$, respectively. Denoting the eigenvalues and eigenvectors of **A** and **B** by $\{(\lambda_i, \mathbf{u}_i), i = 1, \ldots, m\}$ and $\{(\mu_j, \mathbf{v}_j), j = 1, \ldots, n\}$, respectively, it can be verified that the eigenvalues of $\mathbb{K}(\mathbf{A}, \mathbf{B})$ are

$$\{K(\lambda_i, \mu_j), i = 1, \ldots, m, j = 1, \ldots, n\} \tag{9}$$

and the associated eigenvectors are

$$\{\mathbf{u}_i \otimes \mathbf{v}_j, i = 1, \ldots, m, j = 1, \ldots, n\}. \tag{10}$$

The above structure of spectral properties can be readily extended to higher-order Kronecker products.

Two additional definitions are recorded below by way of background. First, the Kronecker product $\mathbf{A} \boxtimes \mathbf{B}$ is termed diagonalizable, if there exists a similarity transformation which renders it equal to $\mathbf{A}' \boxtimes \mathbf{B}'$, where the components of $\mathbf{A}'$ and $\mathbf{B}'$ form a diagonal matrix with reference to a given orthonormal basis. With the aid of (4) and (5) it can be established that $\mathbf{A} \boxtimes \mathbf{B}$ is diagonalizable if and only if both **A** and **B** are diagonalizable. Further, the Kronecker products $(\mathbf{A} \boxtimes \mathbf{B})$ and $(\mathbf{C} \boxtimes \mathbf{D})$ are commuting if $(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{C} \boxtimes \mathbf{D}) = (\mathbf{C} \boxtimes \mathbf{D})(\mathbf{A} \boxtimes \mathbf{B})$. It is clear from (4d) that $\mathbf{A} \boxtimes \mathbf{B}$ and $\mathbf{C} \boxtimes \mathbf{D}$ commute if **A** and **B** commute with **C** and **D**, respectively.

3. EXPONENTIAL REPRESENTATION OF KRONECKER POWERS OF ORTHOGONAL TENSORS

An exponential representation of Kronecker powers of orthogonal tensors is derived here, based on the mathematical properties of Kronecker products outlined in the preceding section, as well as on standard parametrizations of orthogonal second-order tensors. To this end, consider a proper orthogonal second-order tensor $\mathbf{Q}$ in E^3 and recall that it admits the exponential representation

$$\mathbf{Q} = \exp(\theta \mathbf{P}). \tag{11}$$

A simple derivation of (11) is contained in the Appendix. In the above representation, $\mathbf{P}$ is the skew-symmetric second-order tensor whose axial vector $\mathbf{p}$ satisfies $\mathbf{Qp} = \mathbf{p}$, namely $\mathbf{p}$ is an eigenvector of $\mathbf{Q}$ associated with unit eigenvalue. In addition, $\theta \in [0, 2\pi)$ is the rotation angle of $\mathbf{Q}$, such that $\cos\theta \pm i \sin\theta$ are the two complex conjugate eigenvalues of $\mathbf{Q}$, and $i = \sqrt{-1}$.

One of the main results in this article is a general exponential representation of the Kronecker power $\mathbb{Q}_k$ in the form

$$\mathbb{Q}_k = \exp(\theta_k \mathbb{P}), \tag{12}$$

where $\exp(\theta_k \mathbb{P})$ is formally defined as

$$\mathbb{Q}_k = \mathbb{I}_k + \sum_{n=1}^{\infty} \frac{\theta^n}{n!} {}_k\mathbb{P}^n \tag{13}$$

and

$$ {}_k\mathbb{P} = \underbrace{\underbrace{\mathbf{P} \boxtimes \mathbf{I} \boxtimes \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I}}_{k \text{ terms}} + \mathbf{I} \boxtimes \mathbf{P} \boxtimes \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} + \cdots + \mathbf{I} \boxtimes \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} \boxtimes \mathbf{P}}_{k \text{ terms}}. \tag{14}$$

The representation of $\mathbb{Q}_k$ in (12)–(14) has been obtained for the special case $k = 2$ in Mehrabadi *et al.* (1995) by exploiting the particular structure of fourth-order tensors. To prove (12)–(14) for all integers k, first note that

$$\theta_k \mathbb{P} = \mathbb{P}_{(1)} + \mathbb{P}_{(2)} + \cdots + \mathbb{P}_{(k)},$$

where, for any given $i = 1, \ldots, k$,

$$\mathbb{P}_{(i)} = \underbrace{\underbrace{\mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} \boxtimes (\theta \mathbf{P})}_{i \text{ terms}} \boxtimes \mathbf{I} \cdots \boxtimes \mathbf{I}}_{k \text{ terms}}.$$

Using the properties of the Kronecker product in (4) and the definition in (11), it is concluded that

$$\begin{aligned} \exp \mathbb{P}_{(i)} &= \sum_{n=0}^{\infty} \frac{1}{n!} [\mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} \boxtimes (\theta \mathbf{P}) \boxtimes \mathbf{I} \cdots \boxtimes \mathbf{I}]^n \\ &= \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} \boxtimes \sum_{n=0}^{\infty} \left(\frac{\theta^n}{n!} \mathbf{P}^n \right) \boxtimes \mathbf{I} \cdots \boxtimes \mathbf{I} \\ &= \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} \boxtimes [\exp(\theta \mathbf{P})] \boxtimes \mathbf{I} \cdots \boxtimes \mathbf{I} = \mathbf{I} \boxtimes \cdots \boxtimes \mathbf{I} \boxtimes \mathbf{Q} \boxtimes \mathbf{I} \cdots \boxtimes \mathbf{I}. \end{aligned} \tag{15}$$

It is observed from (15) that the exponential of $\mathbb{P}_{(i)}$ is essentially reduced to a Kronecker product form involving the exponential of $\theta\mathbf{P}$. Since the Kronecker products $\mathbb{P}_{(i)}$, $i = 1, 2, \ldots, k$, obviously commute with each other, it follows that the exponential of their sum equals the product of their exponentials. Therefore, with the aid of (4) and (15), it is seen that

$$\begin{aligned}\exp(\theta_k\mathbb{P}) &= \exp\mathbb{P}_{(1)}\exp\mathbb{P}_{(2)}\cdots\exp\mathbb{P}_{(k)}\\ &= (\mathbf{Q}\boxtimes\mathbf{I}\boxtimes\cdots\boxtimes\mathbf{I})(\mathbf{I}\boxtimes\mathbf{Q}\boxtimes\cdots\boxtimes\mathbf{I})\cdots(\mathbf{I}\boxtimes\mathbf{I}\boxtimes\cdots\boxtimes\mathbf{Q})\\ &= \mathbf{Q}\boxtimes\mathbf{Q}\boxtimes\cdots\boxtimes\mathbf{Q} = \mathbb{Q}_k,\end{aligned}$$

which completes the proof.

As background to ensuing developments, it is noted independently of the above exponential representation that, upon applying the product rule of differentiation and recalling eqn (A.7),

$$\frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\theta} = (\mathbf{PQ})\boxtimes\mathbf{Q}\boxtimes\cdots\boxtimes\mathbf{Q}+\mathbf{Q}\boxtimes(\mathbf{PQ})\boxtimes\cdots\boxtimes\mathbf{Q}+\cdots+\mathbf{Q}\boxtimes\mathbf{Q}\boxtimes\cdots\boxtimes(\mathbf{PQ}) = {}_k\mathbb{P}\mathbb{Q}_k, \tag{16}$$

provided that $\mathbf{P}$ does not explicitly depend on θ.

Certain properties of the Kronecker product ${}_k\mathbb{P}$ merit further attention: first, from (4e) and (14), it is observed that ${}_k\mathbb{P}$ is skew-symmetric, in the sense that ${}_k\mathbb{P}+{}_k\mathbb{P}^T = \mathbb{O}$. This, in turn, implies that ${}_k\mathbb{P}$ is normal, i.e. it commutes with its transpose, which guarantees that ${}_k\mathbb{P}$ is also diagonalizable, see Horn and Johnson (1985), Section 2.5, for a proof. Also, computationally convenient recursive formulae for ${}_k\mathbb{P}$ can be established using mathematical induction. In particular, starting with ${}_1\mathbb{P} = \mathbf{P}$,

$${}_{k+1}\mathbb{P} = {}_k\mathbb{P}\boxtimes\mathbf{I}+\mathbb{I}_k\boxtimes\mathbf{P} = \mathbf{P}\boxtimes\mathbb{I}_k+\mathbf{I}\boxtimes{}_k\mathbb{P}, \quad k = 1, 2, \ldots.$$

The above analysis can be readily extended to improper second-order orthogonal tensors and their Kronecker products. Indeed, if $\tilde{\mathbf{Q}}$ is improper, then $\mathbf{Q} = -\tilde{\mathbf{Q}}$ is proper and admits the representation (12). It follows from (4) and (7) that

$$\tilde{\mathbb{Q}}_k = (-1)^k\exp(\theta_k\mathbb{P}),$$

where θ and ${}_k\mathbb{P}$ are defined with reference to the proper orthogonal tensor $\mathbf{Q}$.

4. A GENERALIZED EULER–RODRIGUES FORMULA

The classical Euler–Rodrigues representation of a proper orthogonal tensor in the form

$$\mathbf{Q} = \mathbf{I}+(1-\cos\theta)\mathbf{P}^2+\sin\theta\mathbf{P}$$

can be derived by a series expansion of $\exp(\theta\mathbf{P})$, as shown in the Appendix. The fact that the expansion of $\mathbf{Q}$ contains only three terms can be viewed as a direct consequence of a profound result in linear algebra referred to as Sylvester's interpolation formula, see Horn and Johnson (1991), p. 437. To introduce this result, consider a scalar analytic function f of a real variable x and let $f(\mathbf{A})$ be its associated tensor function, formally defined by a polynomial series expansion, as in Mirsky (1990), Section 11.2.2. Further, assume that $\mathbf{A}$ is a diagonalizable tensor with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ having multiplicities $s_1, s_2, \ldots, s_r$, where $s_i \geqslant 1$ for all $i = 1, 2, \ldots, r$. Then, Sylvester's formula stipulates that

$$f(\mathbf{A}) = P(\mathbf{A}), \tag{17}$$

where $P(x)$ is the unique polynomial of degree $r-1$, which matches $f(x)$ at the r distinct eigenvalues of **A**. Thus, $P(x)$ is written as

$$P(x) = \sum_{i=1}^{r} L_i(x) f(\lambda_i), \tag{18}$$

where $L_i(x), i = 1, 2, \ldots, r$, are the Lagrangian polynomials of degree $r-1$, defined as

$$L_i(x) = \frac{\Pi_{j\neq i}(x-\lambda_j)}{\Pi_{j\neq i}(\lambda_i-\lambda_j)}. \tag{19}$$

Sylvester's formula provides the necessary mathematical framework for deriving a general polynomial representation of $\exp(\theta_k \mathbb{P})$. To this end, start by observing that **P** possesses eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\} = \{0, -i, i\}$, as shown in the Appendix. Therefore, taking into account the definition of ${}_k\mathbb{P}$ in (14), in conjunction with the spectral properties of Kronecker products described in (8)–(10), it follows that the eigenvalues of ${}_k\mathbb{P}$ belong to the set Λ_k specified as

$$\Lambda_k = \{\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}, \quad i_1, i_2, \ldots, i_k = 1, 2, 3\}. \tag{20}$$

Clearly, the above set contains only $2k+1$ distinct eigenvalues and can be rewritten as

$$\Lambda_k = \{-ki, -(k-1)i, \ldots, -i, 0, i, \ldots, (k-1)i, ki\}. \tag{21}$$

Corresponding to each of these distinct eigenvalues, the analytic function $f(x) = \exp(\theta x)$ assumes values

$$\{\cos k\theta - i\sin k\theta, \cos(k-1)\theta - i\sin(k-1)\theta, \ldots, \cos\theta - i\sin\theta, 1,$$
$$\cos\theta + i\sin\theta, \ldots, \cos(k-1)\theta + i\sin(k-1)\theta, \cos k\theta + i\sin k\theta\}. \tag{22}$$

It follows from (18), (19) and (21) that the k-th interpolating polynomial $P_{2k}(x)$ is of degree $2k$. With the aid of (22), the first three members in the hierarchy of these interpolating polynomials are shown to be

$$P_2(\theta, x) = 1 + \sin\theta x + (1-\cos\theta)x^2, \tag{23}$$

$$P_4(\theta, x) = 1 + \tfrac{1}{6}(8\sin\theta - \sin 2\theta)x + \tfrac{1}{12}(15 - 16\cos\theta + \cos 2\theta)x^2$$
$$+ \tfrac{1}{6}(2\sin\theta - \sin 2\theta)x^3 + \tfrac{1}{12}(3 - 4\cos\theta + \cos 2\theta)x^4 \tag{24}$$

and

$$P_6(\theta, x) = 1 + \tfrac{1}{30}(45\sin\theta - 9\sin 2\theta + \sin 3\theta)x$$
$$+ \tfrac{1}{180}(245 - 270\cos\theta + 27\cos 2\theta - 2\cos 3\theta)x^2$$
$$+ \tfrac{1}{24}(13\sin\theta - 8\sin 2\theta + \sin 3\theta)x^3$$
$$+ \tfrac{1}{72}(28 - 39\cos\theta + 12\cos 2\theta - \cos 3\theta)x^4$$
$$+ \tfrac{1}{120}(5\sin\theta - 4\sin 2\theta + \sin 3\theta)x^5$$
$$+ \tfrac{1}{360}(10 - 15\cos\theta + 6\cos 2\theta - \cos 3\theta)x^6. \tag{25}$$

Since it has been previously established that ${}_k\mathbb{P}$ is diagonalizable, Sylvester's formula is applicable and according to (17) yields

$$\mathbb{Q}_k = P_{2k}(\theta, {}_k\mathbb{P}), \quad k = 1, 2, \ldots . \tag{26}$$

Equations (23) and (26) recover the classical Euler–Rodrigues formula, while eqns (24) and (26) result in the representation of $\mathbb{Q}_2$ obtained by a different method in Podio-Guidugli and Virga (1987).

A polynomial representation of the Kronecker powers of improper orthogonal tensors $\tilde{\mathbf{Q}}$ can be determined as a function of the rotation angle θ and the skew-symmetric tensor $\mathbf{P}$ of $\mathbf{Q} = -\tilde{\mathbf{Q}}$ to be

$$\tilde{\mathbb{Q}}_k = (-1)^k P_{2k}(\theta, {}_k\mathbb{P}),$$

as argued in the previous section.

5. APPLICATIONS TO MATERIAL SYMMETRY

5.1. *Structural tensors*

Consider an anisotropic material with symmetry group $\mathscr{S}$. A k-th order tensor ξ is called a structural tensor for this material if

$$\mathbf{Q} \in \mathscr{S} \Leftrightarrow \mathbb{Q}_k \xi = \xi,$$

see, e.g., the article of Zheng and Spencer (1993a) and references therein for further background information. The above identification implies that the structural tensors are eigenvectors of the Kronecker powers of $\mathbf{Q} \in \mathscr{S}$ associated with unit eigenvalues. In light of the general polynomial representation of $\mathbb{Q}_k$ in (26) it is concluded that ${}_k\mathbb{P}$ and $\exp(\theta_k\mathbb{P})$ commute, therefore possess the same eigenvectors. Hence

$${}_k\mathbb{P}\xi = \mathbf{0},$$

which, in turn, implies that the structural tensors of the anisotropic material span the null space of ${}_k\mathbb{P}$, i.e., they are the null eigenvectors of ${}_k\mathbb{P}$. Appealing, again, to the spectral properties of the Kronecker product described in (8)–(10) and recalling (20), it is concluded that the structural tensors are of the general form

$$\mathbf{q}_{i_1} \otimes \mathbf{q}_{i_2} \otimes \cdots \otimes \mathbf{q}_{i_k}, \quad i_1, i_2, \ldots, i_k = 1, 2, 3, \tag{27}$$

where $\mathbf{q}_i$, $i = 1, 2, 3$ are the eigenvectors of $\mathbf{P}$ corresponding to the condition

$$\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k} = 0, \quad i_1, i_2, \ldots, i_k = 1, 2, 3. \tag{28}$$

The eigenvalues and eigenvectors of $\mathbf{P}$ are obtained in the Appendix. Clearly, the complex conjugate eigenvalues of $\mathbf{P}$ must appear in pairs for (28) to hold. Assuming that $k-2r$ zero eigenvalues enter (28) together with r pairs of complex conjugate eigenvalues, it can be readily verified that the total number N of distinct tensor products in the form (27), subject to condition (28), is

$$N = \sum_{r=0}^{s} C_k^{2r} C_{2r}^r, \quad s = \frac{1}{2}(k - \operatorname{mod}(k, 2)),$$

each yielding a different null eigenvector of ${}_k\mathbb{P}$. In the above equation, $C_k^m = [k!/(m!(k-m)!)]$ denotes the binomial coefficient and $\operatorname{mod}(k, m)$ the remainder of the integer division k/m.

The null eigenvectors of ${}_k\mathbb{P}$ are mutually orthogonal, as are the eigenvectors $\mathbf{q}_i, i = 1,2,3$, of $\mathbf{P}$. Referring to the Appendix, write the eigenvectors of $\mathbf{P}$ as $\mathbf{q}_1 = \mathbf{p}$, $\mathbf{q}_2 = \mathbf{q}-i\mathbf{r}$ and $\mathbf{q}_3 = \mathbf{q}-i\mathbf{r}$. Since $\mathbf{q}_2$ and $\mathbf{q}_3$ are complex conjugate, the null eigenvectors of ${}_k\mathbb{P}$ can be always expressed as real vectors by taking suitable linear combinations and eliminating the imaginary parts.

Using the procedure outlined above, the real null eigenvectors of ${}_k\mathbb{P}$ for $k = 1,2$ and 3 are found to be

$$
\begin{aligned}
k = 1: \quad & \mathbf{p}; \\
k = 2: \quad & \mathbf{p}\otimes\mathbf{p}, \quad \mathbf{q}\otimes\mathbf{q}+\mathbf{r}\otimes\mathbf{r}, \quad \mathbf{q}\otimes\mathbf{r}-\mathbf{r}\otimes\mathbf{q}; \\
k = 3: \quad & \mathbf{p}\otimes\mathbf{p}\otimes\mathbf{p}, \\
& \mathbf{p}\otimes(\mathbf{q}\otimes\mathbf{q}+\mathbf{r}\otimes\mathbf{r}), \quad \mathbf{q}\otimes\mathbf{p}\otimes\mathbf{q}+\mathbf{r}\otimes\mathbf{p}\otimes\mathbf{r}, \quad (\mathbf{q}\otimes\mathbf{q}+\mathbf{r}\otimes\mathbf{r})\otimes\mathbf{p}, \\
& \mathbf{p}\otimes(\mathbf{q}\otimes\mathbf{r}-\mathbf{r}\otimes\mathbf{q}), \quad \mathbf{q}\otimes\mathbf{p}\otimes\mathbf{r}-\mathbf{r}\otimes\mathbf{p}\otimes\mathbf{q}, \quad (\mathbf{q}\otimes\mathbf{r}-\mathbf{r}\otimes\mathbf{q})\otimes\mathbf{p}.
\end{aligned}
$$

The preceding derivation is analogous to that in Zheng and Spencer (1993) and the results are in full agreement. It should be noted here that in the case of transversely isotropic materials for which the preferred direction is along $\mathbf{p}$, the proposed method generates the full set of associated k-th order structural tensors.

5.2. *Corotational rates of isotropic tensor functions*

The corotational rate $\mathring{\mathbf{T}}$ of a k-th order tensor $\mathbf{T}$ is an objective tensor defined as the time rate of change of $\mathbf{T}$ measured with reference to a coordinate frame rotating with time-dependent angular velocity $\mathbf{w} = \omega\mathbf{p}$, where $\omega > 0$. The above definition implies that $\mathring{\mathbf{T}}$ is expressed in component form as

$$\mathring{T}_{i_1 i_2 \dots i_k} = \dot{T}_{i_1 i_2 \dots i_k} - W_{i_1 j_1} T_{j_1 i_2 \dots i_k} - W_{i_2 j_2} T_{i_1 j_2 \dots i_k} - \cdots - W_{i_k j_k} T_{i_1 i_2 \dots j_k},$$

where $\dot{T}_{i_1 i_2 \dots i_k}$ are the components of the material time derivative $\dot{\mathbf{T}}$ of $\mathbf{T}$ and $\mathbf{W}$ is the skew-symmetric tensor associated with $\mathbf{w}$. With the aid of (14), the tensor $\mathring{\mathbf{T}}$ is expressed in terms of ${}_k\mathbb{W} = \omega_k\mathbb{P}$ as

$$\mathring{\mathbf{T}} = \dot{\mathbf{T}} - {}_k\mathbb{W}\mathbf{T}. \tag{29}$$

The corotational rate in (29) can be interpreted as the Lie derivative of $\mathbf{T}$ with respect to ${}_k\mathbb{W}$, see Marsden and Hughes (1983), Section 1.6. Indeed, given ${}_k\mathbb{W}$ at time t, and suppressing for brevity the explicit reference to the dependence on t, the flow of ${}_k\mathbb{W}$ at time τ is defined as

$$\mathbb{Q}_k(\tau) = \exp((\tau - t)_k\mathbb{W}), \tag{30}$$

since, with the aid of (16),

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{Q}_k(\tau)\right]_{\tau = t} = {}_k\mathbb{W}.$$

Then, the pull-back $\mathbb{Q}_k^*(\tau)\mathbf{T}(\tau)$ of $\mathbf{T}$ with respect to $\mathbb{Q}_k(\tau)$ is given by

$$\mathbb{Q}_k^*(\tau)\mathbf{T}(\tau) = \mathbb{Q}^{-1}(\tau)\mathbf{T}(\tau) = \mathbb{Q}^T(\tau)\mathbf{T}(\tau),$$

where the orthogonality of $\mathbb{Q}_k$ is invoked. Subsequently, the Lie derivative $\mathscr{L}_{{}_k\mathbb{W}}\mathbf{T}$ of $\mathbf{T}$ with respect to ${}_k\mathbb{W}$ at time t is defined as

$$\mathscr{L}_{k\mathbb{W}}\mathbf{T} = \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{Q}_k^T(\tau)\mathbf{T}(\tau)\right]_{\tau=t}, \tag{31}$$

which, upon using (29), (16) and (30), reduces to $\mathring{\mathbf{T}}$. In the special case where $\mathbf{T}$ is a spatial second-order tensor and ${}_k\mathbb{W}$ is identified with the vorticity tensor, it is immediately seen that $\mathring{\mathbf{T}}$ coincides with the Jaumann rate of T.

Let $\hat{\mathbf{T}}$ be a k-th order isotropic tensor function of tensors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l$ of order $k_1, k_2, \ldots, k_l$, respectively, such that

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l).$$

By the assumption of isotropy,

$$\mathbb{Q}_k\mathbf{T} = \hat{\mathbf{T}}(\mathbb{Q}_{k_1}\mathbf{A}_1, \mathbb{Q}_{k_2}\mathbf{A}_2, \ldots, \mathbb{Q}_{k_l}\mathbf{A}_l), \tag{32}$$

for all the Kronecker powers in (32) generated by an orthogonal second-order tensor $\mathbf{Q}$. It is known that the chain rule applies to the corotational rate of isotropic tensor functions, see Dafalias (1985) and Zheng (1994). In view of (31) and (32), it follows that

$$\begin{aligned}\mathring{\mathbf{T}} &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\{\mathbb{Q}_k^T(\tau)\mathbf{T}(\tau)\}\right]_{\tau=t} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{T}}(\mathbb{Q}_k^T\mathbf{A}_1, \mathbb{Q}_{k_2}^T\mathbf{A}_2, \ldots, \mathbb{Q}_{k_1}^T\mathbf{A}_l)\right]_{\tau=t} \\ &= \frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{A}_1}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\{\mathbb{Q}_{k_1}^T\mathbf{A}_1(\tau)\}\right]_{\tau=t} + \frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{A}_2}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\{\mathbb{Q}_{k_2}^T\mathbf{A}_2(\tau)\}\right]_{\tau=t} + \cdots + \frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{A}_l}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\{\mathbb{Q}_{k_l}^T\mathbf{A}_l(\tau)\}\right]_{\tau=t} \\ &= \frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{A}_1}\mathring{\mathbf{A}}_1 + \frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{A}_2}\mathring{\mathbf{A}}_2 + \cdots + \frac{\partial\hat{\mathbf{T}}}{\partial\mathbf{A}_l}\mathring{\mathbf{A}}_l,\end{aligned}$$

which directly verifies the validity of the above rule.

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APPENDIX: REPRESENTATIONS OF THE ROTATION TENSOR

Define $\mathbf{p}$ as an eigenvector of $\mathbf{Q}$ associated with unit eigenvalue, and let the skew-symmetric tensor $\mathbf{P}$ satisfy

$$\mathbf{P}\mathbf{z} = \mathbf{p} \times \mathbf{z}, \tag{A1}$$

for any vector $\mathbf{z}$ in E^3. It can be easily shown from the above that

$$\mathbf{P}^2 = \mathbf{p} \otimes \mathbf{p} - \mathbf{I}, \quad \mathbf{P}^3 = -\mathbf{P}, \tag{A2}$$

and, upon using mathematical induction, that

$$\mathbf{P}^{2n-1} = (-1)^{n-1}\mathbf{P}, \quad \mathbf{P}^{2n} = (-1)^{n+1}\mathbf{P}^2, \quad n = 1, 2, \ldots. \tag{A3}$$

Also, define unit vectors $\mathbf{q}$ and $\mathbf{r} = \mathbf{p} \times \mathbf{q}$ on a plane normal to $\mathbf{p}$. Taking into account (A1) and (A2) and the definition of $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$, it is observed that

$$\mathbf{P} = \mathbf{r} \otimes \mathbf{q} - \mathbf{q} \otimes \mathbf{r}, \quad \mathbf{P}^2 = -\mathbf{q} \otimes \mathbf{q} - \mathbf{r} \otimes \mathbf{r}. \tag{A4}$$

The first of the two equations in (A4) can be employed to establish that the eigenvalues of $\mathbf{P}$ are $\{0, -i, i\}$ and the associated eigenvectors are $\{\mathbf{p}, \mathbf{q} + i\mathbf{r}, \mathbf{q} - i\mathbf{r}\}$.

The rotation tensor $\mathbf{Q}$ can be expressed by means of the classical Rodrigues formula as

$$\mathbf{Q} = \mathbf{p} \otimes \mathbf{p} + \cos\theta(\mathbf{q} \otimes \mathbf{q} + \mathbf{r} \otimes \mathbf{r}) - \sin\theta(\mathbf{q} \otimes \mathbf{r} - \mathbf{r} \otimes \mathbf{q}). \tag{A5}$$

Consequently, a simple proof of (11) is obtained by writing $\exp(\theta\mathbf{P})$ in series form and taking into account (A3) to get

$$\begin{aligned}
\exp(\theta\mathbf{P}) &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\mathbf{P}^n = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\theta^{2n-1}}{(2n-1)!}\mathbf{P}^{2n-1} + \sum_{n=1}^{\infty} \frac{\theta^{2n}}{(2n)!}\mathbf{P}^{2n} \\
&= \mathbf{I} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\theta^{2n-1}}{(2n-1)!}\mathbf{P} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\theta^{2n}}{(2n)!}\mathbf{P}^2 \\
&= \mathbf{I} + \sin\theta\mathbf{P} + (1 - \cos\theta)\mathbf{P}^2.
\end{aligned} \tag{A6}$$

The equivalence of the right-hand sides in (A5) and (A6) becomes obvious upon recalling (A2) and (A4). In addition, eqns (A3) and (A5) imply that

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\theta} = \cos\theta\mathbf{P} + \sin\theta\mathbf{P}^2 = \mathbf{P}\mathbf{Q}, \tag{A7}$$

provided that $\mathbf{p}$ does not explicitly depend on θ.